GEOMETRY OF THE CASIMIR EFFECT

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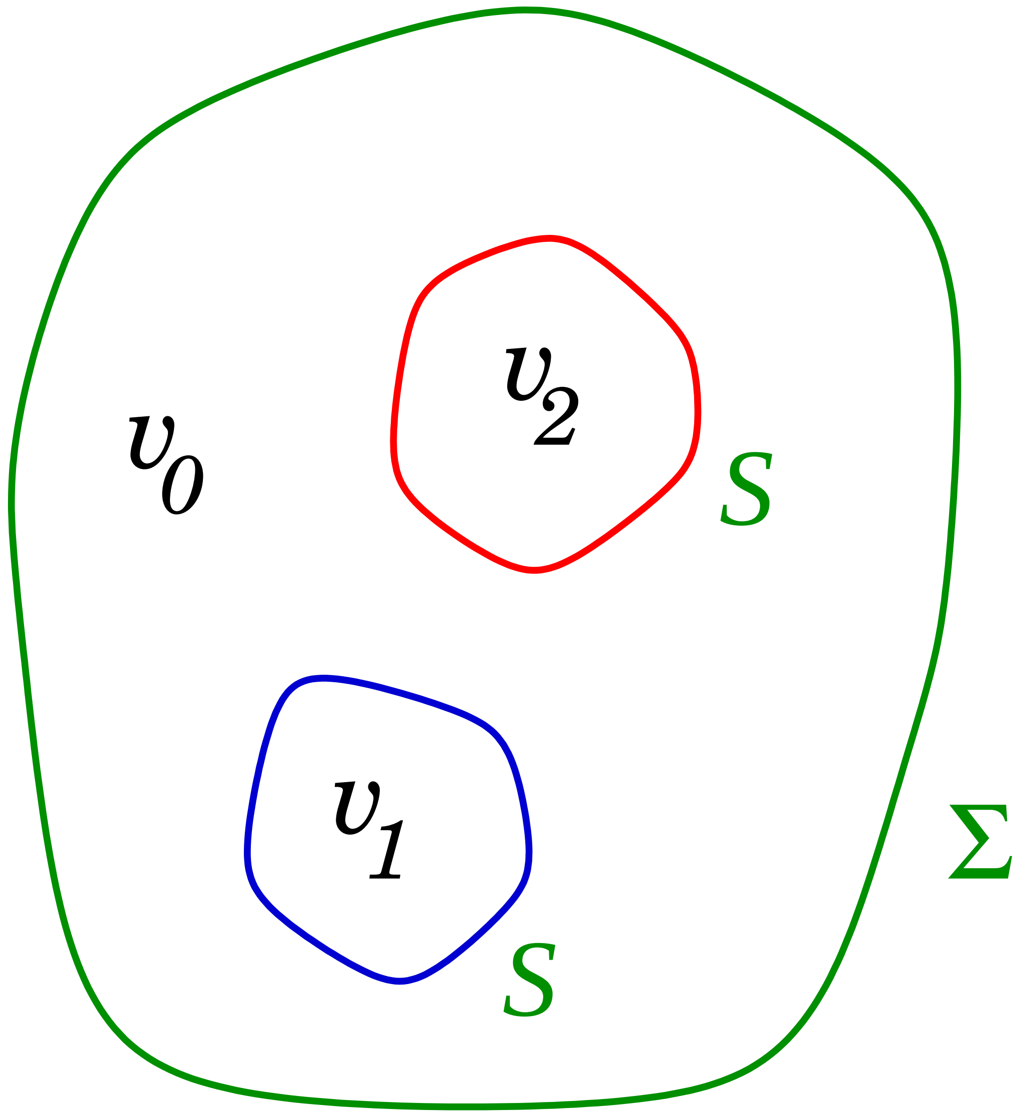
Domain Geometry



Mode Free Energy

For a mode of eigenfrequency cq, the free energy of the quantum oscillator at equilibrium temperature T is

$$f(q) = \varepsilon_0(q) + f_{\mathrm{T}}(q),$$

with

$$\varepsilon_0(q) = \frac{1}{2}\hbar cq, \quad f_{\mathrm{T}}(q) = \beta^{-1}\varphi(\beta\hbar cq),$$

$f_{\mathrm{T}}(q)$ thermal part of the mode free energy, $\beta = 1/k_B T$, with k_B Boltzmann constant, and

$$\varphi(x) = \ln\left(1 - e^{-x}\right), \quad [\varphi(x) \leqslant 0].$$

Free Energy

The eigenspectrum of wavenumbers $q_m^{(v)}$ in a region v is characterized by the *density of modes*

$$\rho^{(v)}(q) = \sum_m \delta(q - q_m^{(v)}) \ ,$$

and the *free energy* of this region is formally equal to

$$\mathcal{F}^{(v)} = \int_0^\infty dq \, \rho^{(v)}(q) \, f(q) \ .$$

If the domain v is infinite, the spectrum is then *continuous*. We imagine the full system to be enclosed in a *large box* Σ, with volume V, which will eventually go to infinity.

Density of Modes

A second difficulty is the fact that the spectra are *not bounded*. For large q, the Eigenmode density has the asymptotic expansion

$$\rho^{(v)}(q) \approx \frac{V}{\pi^2} q^2 - \frac{2}{3\pi^2} \int \frac{d^2\alpha}{R} + \frac{1}{12\pi^2} \int ds \frac{(\pi-\theta)(\pi-5\theta)}{\theta} + O\left(\frac{1}{q^2}\right) + O(q^{5/2} \times \mathrm{osc}) .$$

- *Dominant term* proportional to the *volume* V of the considered region; black-body radiation in the thermodynamic limit
- *Curvature term*, integral over the boundaries of v
- *Wedge term* with a dihedral angle θ along the edge
- *Oscillatory terms* with an amplitude which increases with q

All these terms lead to *divergences*.

Regularization of the Free Energy

- *Infrared divergence* associated with infinite size of vacuum: *box* Σ
- *Ultraviolet divergence* associated with high frequencies $q \to \infty$: *cut-off factor* $\chi(q)$ close to 1 and decreasing sufficiently fast for $q \to \infty$ so as to restore convergence of the integral

Partition of the whole space (within enclosure Σ) into a set of connected regions v, some of which coincide with the actual vacuum, the other ones are the interiors of the conductors:

Regularized free energy

$$\mathcal{F}_{\text{reg}} = \int_0^\infty dq \left[\sum_v \rho^{(v)}(q) - \rho^{(\Sigma)}(q) \right] f(q)\chi(q) \equiv \int_0^\infty dq\, \delta\rho(q)\, f(q)\chi(q)$$

Renormalization of the Free Energy

$\mathcal{F}_{\text{reg}}$ has (in the absence of wedges) a *finite limit* $\tilde{\mathcal{F}}$ as $\Sigma \to \infty$ and $\chi(q) \to 1$, independently of the shapes of Σ and $\chi(q)$.

Renormalized Casimir Energy

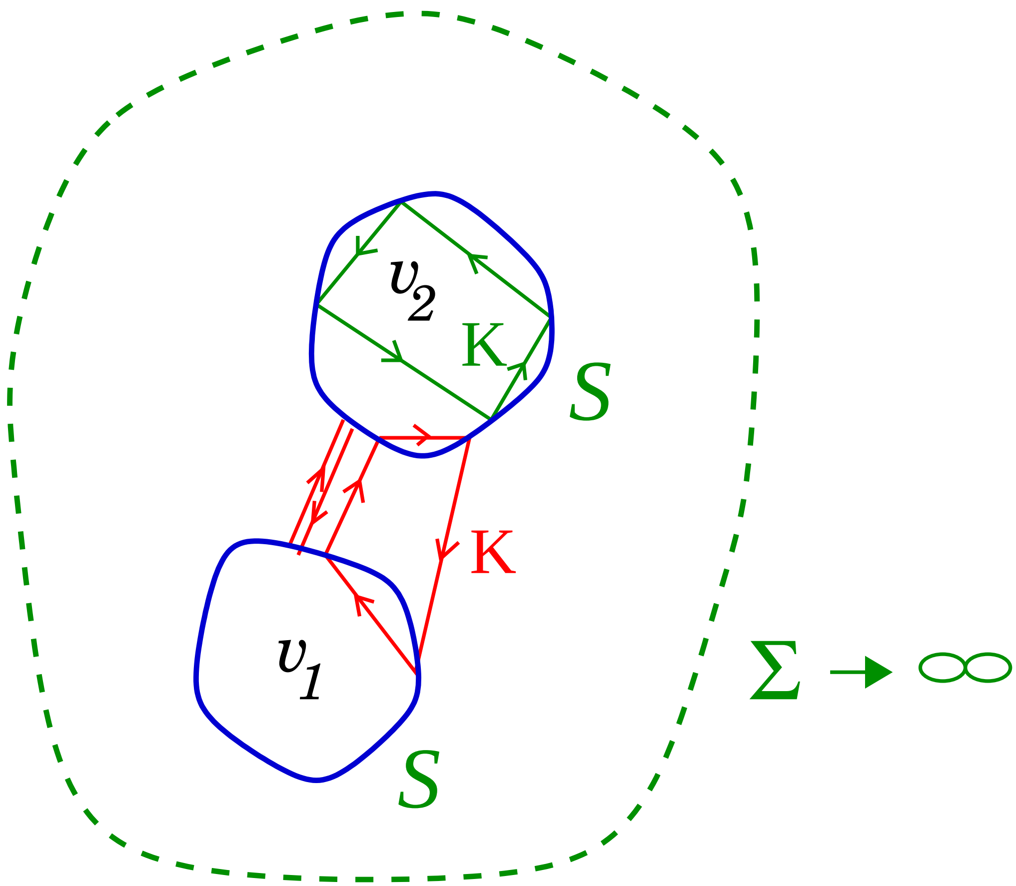
Altogether, one finds the limit of the renormalized Casimir energy for the exterior surface $\Sigma \to \infty$ and cut-off function $\chi(q) \to 1$ as (*R. Balian & B.D.* (1978))

$$\tilde{\mathcal{E}}_0 = \frac{\hbar c}{\pi} \int_0^\infty dy [\Psi(y) - \Psi(\infty)],$$

where the function $\Psi(y)$ encapsulates the *geometry* of the boundaries S, with $y = \mathrm{Im}\, q$.

The *Casimir energy* i.e., *the variation of the zero-point energy* induced by the introduction of the perfectly conducting shells, is the product of $\hbar c$ by a factor with dimension L^{-1} depending on the shape of S.

Multi-Scattering Expansion



Characteristic Function Ψ

A closed expression for the function $\Psi(y)$ is

$$\Psi(y) = -\frac{y}{4}\frac{d}{dy}\mathrm{Tr}\ln(1-K^2) \ ,$$

where the trace Tr and the products stand for integration over boundaries S of surface points $\alpha, \beta \in S$ and summation over tensor indices of K:

$$K(\alpha,\beta;y) = \frac{1}{2\pi} n_\alpha \wedge \nabla_\alpha \wedge \left[\frac{e^{-y(\alpha-\beta)}}{|\alpha-\beta|}\mathbf{1}\right] \ ,$$

where n_α is the normal at point α on surface S. K decreases exponentially at large distances. The series obtained by expanding $\ln(1-K^2)$ in powers of K^2 converges.

Renormalized Thermal Free Energy

Thermal part of the renormalized Casimir free energy

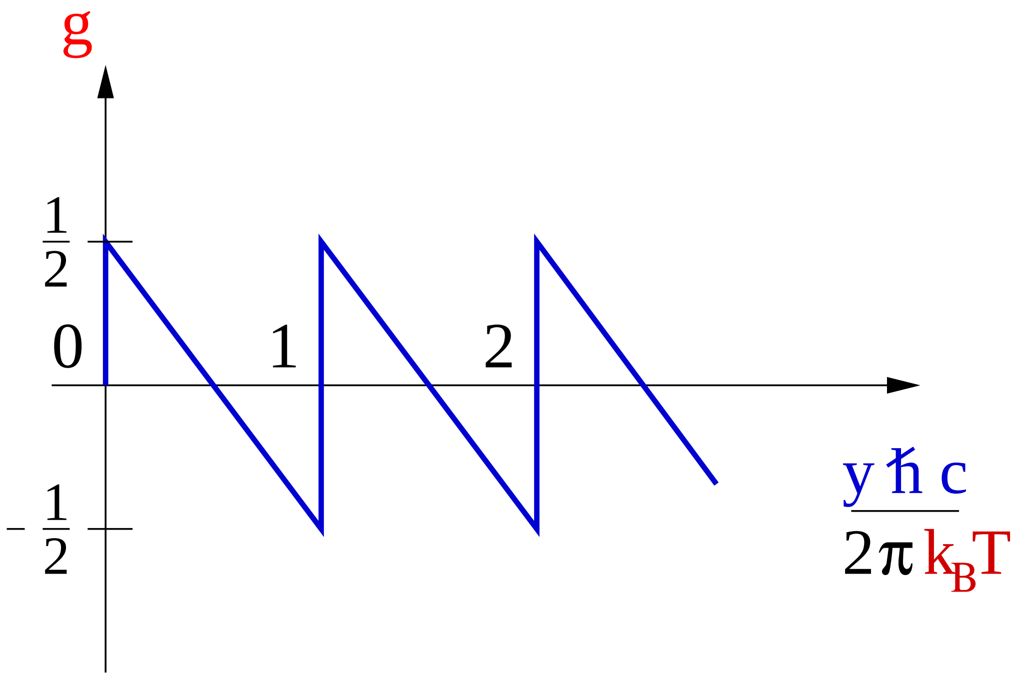
$$\tilde{\mathcal{F}}_{\mathrm{T}} = 2k_B T \int_0^\infty \frac{dy}{y} \left[\Psi(y) - \Psi(+0)\right] g(y) \ ,$$

where the *temperature* appears through the sawtooth function

$$g(y) = \frac{1}{2} - \frac{y}{\eta} + \sum_{n=1}^{\infty} \theta(y - n\eta) \ , \ \eta = \frac{2\pi k_B T}{\hbar c} \ ,$$

with $\theta(x) = 0$ for $x < 0$, $\theta(x) = 1$ for $x > 0$.

Thermal g Function



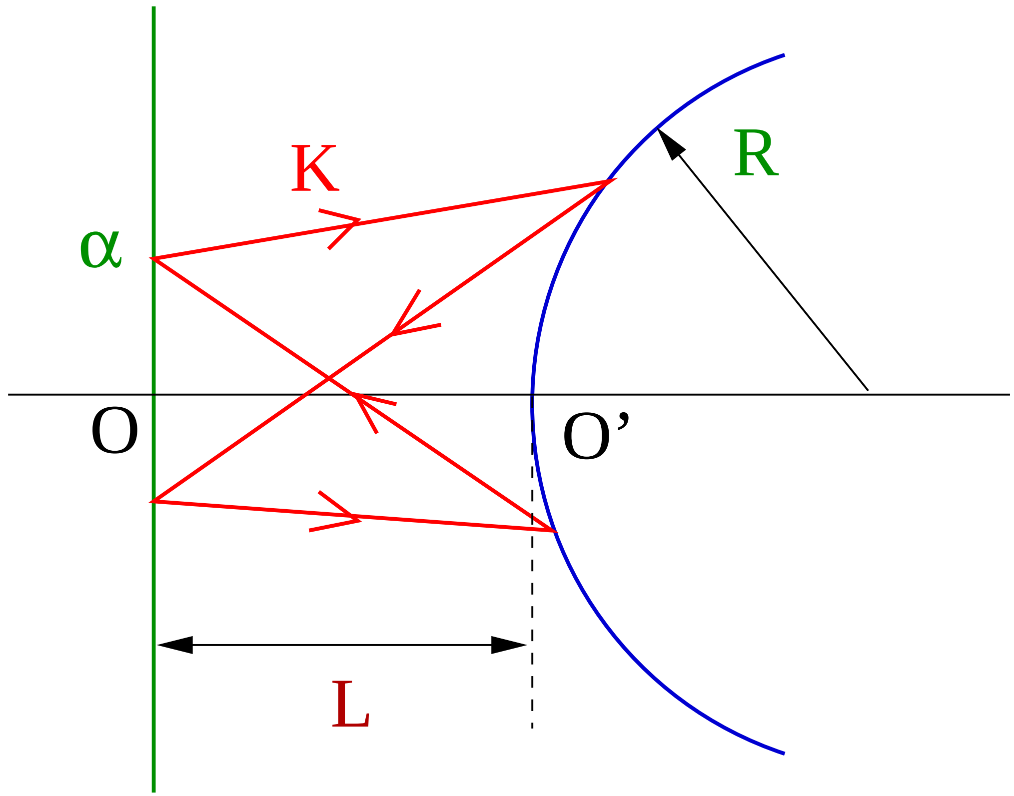
Ψ for Parallel Plates

For two parallel plates with area S, lying a distance L apart:

$$\Psi(y) = \frac{Sy^2}{2\pi} \ln(1 - e^{-2yL}) \ ,$$

yielding the original Casimir result at $T = 0$.

Multi-Scattering Expansion for a Plane and a Sphere



Derjaguin Approximation Revisited

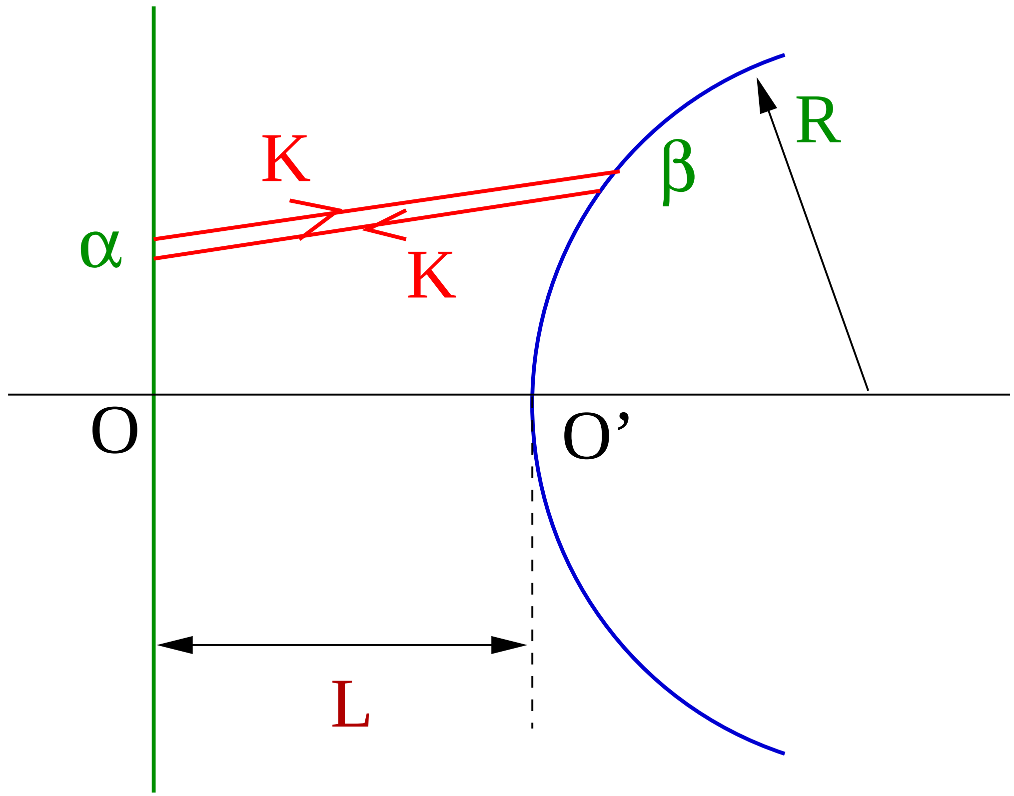
For *neighbouring conductors*, the *multiple scattering expansion* can be used to justify *Derjaguin's approximation*.

Consider a plane and a sphere with radius R, and let L be their shortest distance. One starts from a point α of the plane:

$$\Psi(y) = -\frac{1}{4}\int d^2\alpha\, y\frac{d}{dy}\left[\mathrm{tr}\ln(1-\mathrm{K}^2)\right]_{\alpha\alpha} ,$$

where the trace tr is on the tensor index only. Suppose $L \ll R$. Owing to the exponential decrease of K, the integral is dominated by the contributions such that α lies at a distance x of order L from the sphere, and all successive scattering points also lie at a distance of order L from α. The integrand is approximately the same as for two parallel plates lying at a distance x apart, given by $\pi^{-1}y^2\ln(1-e^{-2yx})$. This is just Derjaguin's approximation.

Two-Scattering Approximation



Two-Scattering Approximation

A useful approximation is the *two-scattering approximation*. One retains for $\Psi(y)$ only the lowest order term $\frac{1}{2}\mathrm{Tr}KydK/dy$:

$$\Psi^{(2)}(y) = 2y\frac{d}{dy}\int d^2\alpha d^2\beta \frac{dG_0(|\alpha-\beta|)}{dn_\alpha}\frac{dG_0(|\alpha-\beta|)}{dn_\beta}$$

$$= -\frac{y^2}{8\pi^2}\int d^2\alpha d^2\beta (n_\alpha.\rho)(n_\beta.\rho)\frac{1}{\rho}\frac{d}{d\rho}\frac{e^{-2y\rho}}{\rho^2}$$

where ρ is the vector $\alpha-\beta$. Numerical tests show that this approximation should be fairly good; for example it yields for the Casimir force at $T=0$ between two parallel plates the correct result times $90/\pi^4$, an error of 8%.

Plane-Sphere Casimir Energy, Two-Scattering Approximation

Casimir energy at $T = 0$, for $L \ll R$,

$$\mathcal{E}_0^{(2)} \approx -\frac{\hbar c R}{8\pi L^2} + \frac{\hbar c}{8\pi L} \; ,$$

which exhibits a correction in L/R to the two-scattering Derjaguin contribution.

Long-Range Interactions

The Casimir forces between conductors lying *far apart* can be evaluated by means of the free energy. Such forces have thus the same nature as *van der Waals forces*, except for the retarded character of the interaction. Two conductors at a large distance L apart *attract* each other as $1/L^8$ at zero temperature, as T/L^7 at high temperature. *Torques* are also found for anisotropic bodies. The same results hold for a conducting body facing a mirror, which is attracted by it.

Wrinkling Effects

The existence of constraints that tend to wrinkle conducting surfaces at high temperature is confirmed by the study of *small deformations of a thin foil*.

Constraints tend to create ripples with wavelengths larger than $2.9\hbar c/T$, and to restore flatness for smaller wavelengths. The Casimir effect proper, at zero temperature, tends to suppress curvature. A *conducting plane foil* is *stable at* $T = 0$, but *unstable at* $T \neq 0$ under small deformations.

Spherical Shell

For a *spherical shell* S with radius R the Casimir energy at $T = 0$ is [Boyer, 1969, R.Balian & B.D., 1978]

$$\mathcal{E}_0 = 0.046\hbar c/R \; .$$

At high temperature, we find as expected

$$\tilde{\mathcal{F}}_{\mathrm{T}} = -\frac{k_B T}{4}\left[\ln(k_B T R/\hbar c) + 0.769\right] + \cdots$$

Usually, attractive Casimir forces. These forces at any temperature *expand the sphere* and increase with T. The radiation pressure exerted from inside thus exceeds that exerted from outside, contrary to parallel plates.

Conclusion

In conclusion, in these experiments one thus observes the *macroscopic* electromagnetic force generated by vacuum quantum fluctuations, proportional to $\hbar c$, and this in the absence of any charge and any (transverse) photon in the cavity! Planck, when inventing his famous black-body formula, even without imagining the extra vacuum energy term, really had lifted a corner of the veil.