

CASIMIR EFFECT, ZETA FUNCTIONS, AND COSMOLOGY

Casimir effect and the Fulling-Davies theory

The 1948 paper by H.B.G. Casimir [1] was the beginning of a whole branch of research which aims at answering, both from a fundamental and from a practical perspective, very profound questions on the vacuum structure of Quantum Field Theory. The interest of the subject is certified by the impressive number of papers published in recent years (just for a choice see [2, 3] and references therein). To wit, fluctuations of the quantum vacuum appear practically everywhere, the key issue being to ascertain if these contributions are relevant or not for the physical (chemical or biological) process in question. They have been argued to be irrelevant for sonoluminescence (a 10^{-5} contribution), but they are very important for accurate calculations in laser cavities, wetting phenomena of alkali compounds by Helium-3, the sticking of small plates in MEM and NEM devices, and so on [2, 3].

When the plates move quickly, as with a high-frequency vibration, one has a dynamical Casimir effect, a phenomenon studied by S. Fulling and P. Davies in 1976 [4]. Actually, moving mirrors further modify the structure of the quantum vacuum, what manifests in the creation and annihilation of particles. Once the mirrors return to rest, a number of the produced particles may still remain which can be interpreted as radiated particles. This flux has been calculated in several situations by different procedures (averaging over fast oscillations multiple scale analysis, with the rotating wave approximation, numerical techniques, etc.). For a single, perfectly reflecting mirror, the number of produced particles as well as their energy diverge while the mirror moves. Several renormalization prescriptions had been devised to obtain a well-defined energy, however, for some trajectories this finite energy turned not to be a positive quantity and could not be identified with the energy of the produced particles (see e.g. [4]), a situation that was resolved with a new approach [5], which relies upon two rather simple ingredients: (i) proper use of a Hamiltonian method and (ii) the consideration of partially transmitting mirrors, which become transparent at sufficiently high frequencies. Realistic mirrors will always satisfy this condition and, in this way, it can be proven both that the number of created particles is finite and also that their energy is always positive for the whole trajectory corresponding to the mirrors' displacement. Then the energy of the field at any time t is equal, with the opposite sign, to the work performed by the reaction force up to time t . Such force is usually split into two parts: a dissipative force whose work equals minus the energy of the particles that remain, and a reactive force vanishing when the mirrors return to rest. It was shown in [5] that the radiation-reaction force calculated from the Hamiltonian approach for partially transmitting mirrors satisfies, all the way the energy conservation law and can thus naturally account for the creation of positive energy particles. Also, the dissipative part we obtain agrees with the one calculated by other methods, as using the Heisenberg picture or other effective Hamiltonians. This is basically the result of a proper, physically meaningful renormalization prescription and shows the importance of using rigorous mathematical procedures in the description of physical phenomena.

On the method of zeta-function regularization

Many fundamental calculations of QFT reduce, in essence, to the computation of the determinant of some suitable operator: at one-loop order, any such theory reduces in fact to a theory of determinants. The operators involved are pseudodifferential (Ψ DO), in loose terms 'some analytic functions of differential operators' (such as $\sqrt{1 + D}$ or $\log(1 + D)$, but *not* $\log D$). This is explained in detail in [6]. These determinants involve in its definition a regularization (being related to operators that are not trace-class). This piece of calculus falls outside the scope of the standard disciplines and has many things in common with divergent series theory, but lacks any reference comparable to the book of Hardy [7]. Actually, this question was already addressed by Weierstrass in a way not without problems, since it leads to non-local contributions that cannot be given a physical meaning in QFT. For completion, let us mention the well established theories of determinants for degenerate operators, for trace-class operators in the Hilbert space, Fredholm operators, etc. [8].

Hawking introduced zeta-function regularization [9] as a tool to deal with infinities in QFT in curved spacetime.[10, 11, 12] One could try to deal with Quantum Gravity using the canonical approach, by defining

an arrow of time and working on the space-like hypersurfaces perpendicular to it, with equal time commutation relations, but: (i) There are many topologies of the space-time manifold that are not a product $\mathbf{R} \times M_3$. (ii) Such non-product topologies are sometimes very interesting. (iii) What does it mean ‘equal time’ in the presence of Heisenberg’s uncertainty principle?

One is thus naturally led to path-integrals: $\langle g_2, \phi_2, \mathcal{S}_2 | g_1, \phi_1, \mathcal{S}_1 \rangle = \int \mathcal{D}[g, \phi] e^{iS[g, \phi]}$, where g_j denotes the spacetime metric, ϕ_j are matter fields, \mathcal{S}_j general spacetime surfaces ($\mathcal{S}_j = M_j \cup \partial M_j$), \mathcal{D} a measure over all possible ‘paths’ leading from the $j = 1$ to the $j = 2$ values of the intervening magnitudes, and S is the action: $S = \frac{1}{16\pi G} \int (R - 2\Lambda) \sqrt{-g} d^4x + \int L_m \sqrt{-g} d^4x$, R being the curvature, Λ the cc, g the determinant of the metric, and L_m the Lagrangian of the matter fields. Stationarity of S under the BCs $\delta g|_{\partial M} = 0$, $\vec{n} \cdot \vec{\partial} \delta g|_{\partial M} = 0$, leads to Einstein’s equations: $R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab} = 8\pi G T_{ab}$, T_{ab} being the energy-momentum tensor of the matter fields, $T_{ab} = \frac{1}{2\sqrt{-g}} \frac{\delta L_m}{\delta g^{ab}}$. The path-integral formalism provides a way to deal ‘perturbatively’ with QFT in curved spacetime backgrounds.[11] First, through a rotation in the complex plane one defines an Euclidean action: $iS \rightarrow -\hat{S}$. One can also easily introduce the finite temperature formalism by the substitution $t_2 - t_1 = i\beta$, which yields the partition function $Z = \sum_n e^{-\beta E_n}$. If one now adheres to the principle that the Feynman propagator is obtained as the limit for $\beta \rightarrow \infty$ of the thermal propagator, we have shown [13] that the usual principal-part prescription in the zeta-function regularization method need *not* be imposed any more as an additional assumption, since it follows from this more general and natural principle. Next comes the stationary phase approach (also called one-loop or WKB), for calculating the path integral, which consists in expanding around a fixed background: $g = g_0 + \bar{g}$, $\phi = \phi_0 + \bar{\phi}$, and leads to the following expansion in the Euclidean metric: $\hat{S}[g, \phi] = \hat{S}[g_0, \phi_0] + S_2[\bar{g}, \bar{\phi}] + \dots$ This is most suitably expressed in terms of determinants (for bosonic, resp. fermionic fields) of the kind (here A, B are the relevant (pseudo)differential operators in the corresponding Lagrangian): $\Delta_\phi = \det \left(\frac{1}{2\pi\mu^2} A \right)^{-1}$, $\Delta_\psi = \det \left(\frac{1}{2\mu^2} B \right)$, optimally computed with zeta techniques.

For its application in practice [14], the method of zeta regularization relies on the existence of simple formulas that give the analytic continuation of $\zeta(s)$ from the region of the complex plane extending to the right of the abscissa of convergence, $\text{Re } s > s_0$, to the rest of it [15, 16]. These are not only the reflection formula, but also some other expressions, as Jacobi’s theta function identity, Poisson’s and Plana’s resummation formulae, and the Chowla-Selberg formula. But some of these expressions are often restricted to specific cases, and their explicit derivation is usually involved [16]. A fundamental property shared by all zeta functions is the existence of a reflection formula (called the functional equation by mathematicians). For the Riemann zeta function: $\Gamma(s/2)\zeta(s) = \pi^{s-1/2}\Gamma(1-s/2)\zeta(1-s)$. For a generic zeta function, $Z(s)$, we may write it as: $Z(\omega - s) = F(\omega, s)Z(s)$. It allows for its analytic continuation in an easy way —what is, in the most simple cases, the whole story of the zeta function regularization procedure. But the analytically continued expression thus obtained is just another series, which has again a slow convergence behavior, of power series type (the same of the original series, on its convergence domain). S. Chowla and A. Selberg found a formula for the Epstein zeta function in two dimensions, that yields *exponentially* quick convergence *everywhere*. In [17] a first attempt was done to try to extend it to inhomogeneous forms; later, to higher dimensions [6, 18], both for the homogeneous (quadratic form) and non-homogeneous (quadratic plus affine form) cases. However, some of the new formulas (remarkably the ones corresponding to the zero-mass case, e.g., the original CS framework) were not explicit, and involved solving a non-trivial recurrence, that was solved in [19] and *explicit* formulas were obtained. Aside from the quadratic case, the linear one is also important for its many applications (system of harmonic oscillators or a multidimensional one). The most general linear zeta function studied is the Barnes’ one. Again, many explicit expressions are missing here, as for its derivative in the general case.

Assume the Hamiltonian operator, H , has a spectral decomposition of the form (think of a quantum harmonic oscillator): $\{\lambda_i, \varphi_i\}_{i \in I}$, being I some set of indices (which can be discrete, continuous, mixed, multiple, ...). Then, the quantum vacuum energy is obtained as: [16]

$$E/\mu = \sum_{i \in I} \langle \varphi_i, (H/\mu) \varphi_i \rangle = \text{Tr}_\zeta H/\mu = \sum_{i \in I} \lambda_i/\mu = \sum_{i \in I} (\lambda_i/\mu)^{-s} \Big|_{s=-1} = \zeta_{H/\mu}(-1), \quad (1)$$

where ζ_A is the zeta function corresponding to the operator A , and the equalities are in the sense of analytic

continuation (generically, A is not of the trace class).¹ Note that the formal sum over the eigenvalues is usually ill defined, and that the last step involves analytic continuation. A regularization parameter μ with dimensions of mass appears in the process, to render the eigenvalues dimensionless, so that the zeta function can be defined! We shall not discuss these basic details here, which are just at the starting point of the whole renormalization procedure).

Quantum vacuum fluctuations and cosmology

Our universe appears to be spatially flat and to possess a non-vanishing cosmological constant (cc), as introduced by Einstein in his equations, and with the same sign. By the equivalence principle the vacuum expectation value of the stress-energy tensor $\langle T_{\mu\nu} \rangle \equiv -\mathcal{E}g_{\mu\nu}$ appears on the rhs of Einstein's equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi G(\tilde{T}_{\mu\nu} - \mathcal{E}g_{\mu\nu}). \quad (2)$$

It affects cosmology: $\tilde{T}_{\mu\nu}$ contains excitations above the vacuum, and is thus equivalent to a cosmological constant $\Lambda = 8\pi G\mathcal{E}$. Recent observations yield [21]

$$\Lambda_{\text{obs}} = (2.14 \pm 0.13 \times 10^{-3} \text{ eV})^4 \sim 4.32 \times 10^{-9} \text{ erg/cm}^3$$

This idea is old and goes back to Zel'dovich [22], who already stated that the cc gets contributions from the zero point fluctuations

$$E_0 = \frac{\hbar c}{2} \sum_n \omega_n, \quad \omega = k^2 + m^2/\hbar^2, \quad k = 2\pi/\lambda. \quad (3)$$

Contributions of vacuum fluctuations will produce a divergent quantity (integral over all possible frequencies). However, nobody will trust present QFT beyond the Planck scale, thus the Planck frequency poses a natural cut off. Evaluating in a box and putting such cut-off at k_{max} corresponding to the Planck value

$$\rho \sim \frac{\hbar k_{\text{Planck}}^4}{16\pi^2} \sim 10^{123} \rho_{\text{obs}}. \quad (4)$$

The issue of the cc has got renewed thrust from the observational evidence of an acceleration in the expansion of our Universe, initially reported by two different groups [23]. As a consequence, many theoreticians have urged to try to explain this fact, and also to try to reproduce the precise value of the cc coming from these observations, but it is more difficult to explain why the cc is so small but non-zero, than to build theoretical models where it exactly vanishes [25]. Many different ideas have appeared to solve or at least ameliorate this problem. A very nice one involves the number of species [24].

Before going on, it is interesting to observe that the Casimir effect dates back from the very same year, 1948, as the discovery of renormalized QED. Curiously enough, however, Feynmann is reported to have argued that vacuum fluctuations could not be affected by gravity. Thus, a very basic issue is if one is right in writing the equations above and thus assuming that the vacuum energy satisfies the equivalence principle of GR. In other words, how the renormalized Casimir energy of a pair of plates couples to gravity? The answer to this question is less straightforward than one might suspect. In fact very recently disparate answers have appeared in the literature: forces that depend on the orientation of the Casimir apparatus wrt the gravitational field of the earth, etc. Essentially there are two ways to proceed with the calculations. A gauge-invariant procedure: as the energy-momentum tensor of the physical system must be conserved, one needs to include a physical mechanism holding the plates apart against the Casimir force, what leads in practice to a very complicated model-dependent calculation [26]. The alternative, and more practical, way is to find a physically natural coordinate system, more realistic than any other. A reasonable one, used for these purposes, is the Fermi coordinate system, a general-relativistic extrapolation of the concept of an inertial coordinate frame. This has been used in [26], a paper initially criticized in [27] (again, the reason one gets different answers in different coordinate systems is that the starting point is not gauge-invariant).

¹Warning: this ζ -trace is actually no trace in the usual sense. In particular, it is highly non-linear, as often explained elsewhere [20]. Many important mistakes and erroneous conclusions have been the result of ignoring this.

However, in the end agreement seems to have been reached (at least in Fermi and Rindler coordinates), after renormalizing the mass of the plates, that the Casimir energy contributes with a gravitational mass in accordance with the equivalence principle. The end word is that, although a rigorous, gauge-invariant proof is still lacking, the calculations performed until now hint towards the positive answer.

A cosmological imprint of the Casimir effect?

Having clarified the important issue above, now, rather than trying to understand the fine-tuned cancelation of the enormous contributions mentioned at the *local* level, one can think of other more simple ideas, as those related with the topology of the universe [28] at large (global topology, braneworld models) and at small (compactified additional dimensions), in connection with the possibility that fields of very small mass pervading the universe (in addition to neutrinos) probably exist (axions, wimps). They are indeed ubiquitous in inflationary models, quintessence theories, etc. A possible strategy at solving the new cc problem is two step: first look back to the old problem and try to prove that the cc is essentially zero, say for a given (flat) topology (or other ideal symmetries) and, second, identifying some extra (small) contributions, of the order of magnitude corresponding to the recent observations, coming from the fact that the universe has in fact a different topology (or the ideal symmetries are slightly broken). In this way the calculation of the cc is a Casimir-like calculation since it corresponds to the difference between two situations. Of course the hard part continues to be to show that the ‘bulk’ contribution is vanishing (the old cc problem). What has been addressed in a number of papers is, so to say, the ‘perturbative part’ of the *new cc* problem [29].

The most simple model of this sort assumes the existence of both large and small dimensions (the total number of large spatial coordinates being always three), some of which may be compactified, so that the global topology of the universe plays an important role. There is an extensive literature both on what is the global topology of (spatial sections) of the universe and also on possible contribution of the Casimir effect as a source of some sort of cosmic energy, e.g., as in the case of the creation of a neutron star. Arguments favor different topologies, as a compact hyperbolic manifold, for the spatial section, that could have clear observational consequences. At a second stage it will have sense to consider all possibilities concerning the nature of the fields, the different models for the topology of the universe, and the possible BCs, with their effects on the sign of the force too. Zeta function regularization techniques have been successfully used to do the calculations [30].

A crucial issue is the *sign* of the resulting force. For scalar fields and the usual compactifications or BCs do not seem possible to get the right sign corresponding to the accelerated expansion of the universe. However, in worldbrane models and others, involving supergravitons and fermion fields, one has been able to prove that the appropriate sign can be obtained under particular but quite natural conditions. Specifically, in the case of the torus topology we have obtained [33] that the topological contributions to the effective potential have in fact a fixed sign, which depends on the BC one imposes. It is negative for periodic fermionic fields in a supersymmetric theory, and positive for anti-periodic fields. Thus, topology provides a mechanism which, in a most natural way, permits to have a positive cc in a multi-supergraviton model with anti-periodic fermions [33]. Moreover, the value of the cc is regulated by the corresponding size of the torus (as is easy to see in the scalar case above). One can most naturally use in this case the minimum number, $N = 3$, of copies of bosons and fermions, and show that the order of magnitude of the observational values for the cc can be reproduced invoking quite natural assumptions. However, a complete, convincing explanation of the observed cc as coming from a Casimir effect is still lacking.

Another approach we have followed [34] (and those are already results directly coming from CASIMIR Research Network collaborations), just involving for the moment scalar fields, deals with the Casimir energy and force for a massive field with general curvature coupling parameter, subject to Robin boundary conditions on two codimension-one parallel plates, located on a $(D + 1)$ -dimensional background spacetime with an arbitrary internal space. The most general case of different Robin coefficients on separate plates have been considered there. With independence of the geometry of the internal space, the Casimir forces are seen to be attractive for special cases of Dirichlet or Neumann boundary conditions on both plates and repulsive for Dirichlet boundary condition on one plate and Neumann boundary condition on the other. For Robin boundary conditions, the Casimir forces can be either attractive or repulsive, depending on the Robin coefficients and the separation between the plates, what is actually remarkable (and useful). Indeed, we have

demonstrated the existence of an equilibrium point for the interplate distance, which is stabilized due to the Casimir force, and shown that stability is enhanced by the presence of the extra dimensions. Applications of these properties in braneworld models were given and the corresponding results were generalized to the geometry of a piston with arbitrary cross section. Recently [35] we have considered a massive scalar field with an arbitrary curvature coupling parameter in the region between two infinite parallel plates on background of de Sitter spacetime. The field is prepared in the Bunch-Davies vacuum state and is constrained to satisfy Robin boundary conditions on the plates. For the calculation, a mode-summation method has been used, supplemented with a variant of the generalized Abel-Plana formula. This allows to explicitly extract the contributions to the expectation values which come from each single boundary, and to expand the second-plate-induced part in terms of exponentially convergent integrals. Several limiting cases of interest have been studied. The Casimir forces acting on the plates have been evaluated, and it has been seen that the curvature of the background spacetime decisively influences the behavior of these forces at separations larger than the curvature scale of de Sitter spacetime. In terms of the curvature coupling parameter and the mass of the field, two very different regimes are realized, which exhibit monotonic and oscillatory behavior of the vacuum expectation values, respectively. The decay of the Casimir force at large plate separation is shown to be power-law (monotonic or oscillating), with independence of the value of the field mass. A motivation for studying these systems in cosmology is that if the universe, as it seems, is going to accelerate for ever, standard cosmology will lead asymptotically to a dS universe. Another motivation is related to the holographic duality known to hold between quantum gravity on dS spacetime and a quantum field theory living on its boundary, identified with the timelike infinity surface of the dS spacetime. In summary, this simplified set up contains some basic ingredients that more full-fledged cosmological models will necessarily have to incorporate. We are on the way to construct those, relying all the time on the most recent and accurate observational data.

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